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The mixmaster universe model, revisited

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Abstract. We re-examine the singularity structure of the mixmaster universe model based on recent developments concerning the treatment of negative resonances. In a previous publication we have shown that this model satisfied the Ablowitz–Ramani–Segur test for the Painlevé property. Now using a different method, called the 'perturbative Painlevé approach' which involves series in both positive and negative powers of the independent variable, we find that *some compatibilities related to negative resonances are not satisfied. This leaves the possibility* open that the mixmaster model may be non-integrable.

The mixmaster universe (or Bianchi IX) model [1, 2] has been analysed recently [3] using the techniques of singularity analysis. This study was motivated by a controversy that persists concerning the dynamical properties of this model. Initially, the mixmaster universe model was proposed as a model of the early universe and its behaviour was, supposedly, ergodic and mixing [4, 5]. However, when accurate calculations of the Liapunov characteristic numbers (LCN) were performed [6–9], the maximal LCN was found to be equal to zero, thus indicating the absence of chaotic behaviour. Although the calculations of LCN are particularly delicate in this case, these results suggested the possibility that the mixmaster universe model might not be chaotic at all. We have thus examined [3] the model using a well known integrability predictor, namely the Painlevé test, in the form of the Ablowitz–Ramani–Segur (ARS) algorithm [10]. The interesting result of this study, corroborated by the application in [11] of Ziglin's approach [12], is that no indication of non-integrability emerged.

The equations of motion for the mixmaster universe model are usually written as [13]

$$\begin{aligned} 2\ddot{\alpha} &= (e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha} \\ 2\ddot{\beta} &= (e^{2\gamma} - e^{2\alpha})^2 - e^{4\beta} \\ 2\ddot{\gamma} &= (e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma} \end{aligned} \tag{1}$$

with the supplementary condition

$$4(\dot{\alpha}\dot{\beta} + \dot{\beta}\dot{\gamma} + \dot{\gamma}\dot{\alpha}) = e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2e^{2(\alpha+\beta)} - 2e^{2(\beta+\gamma)} - 2e^{2(\gamma+\alpha)} \tag{2}$$

where dots denote derivatives with respect to time τ. If we introduce the (non-canonical) variables

$$\begin{aligned} &X = e^{2\alpha} \qquad Y = e^{2\beta} \qquad Z = e^{2\gamma} \\ &p_x = -(\dot{\beta} + \dot{\gamma}) \qquad p_y = -(\dot{\gamma} + \dot{\alpha}) \qquad p_z = -(\dot{\alpha} + \dot{\beta}) \end{aligned} \tag{3}$$

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we find

$$\dot{X} = X(p_x - p_y - p_z) \qquad \dot{Y} = Y(p_y - p_z - p_x) \qquad \dot{Z} = Z(p_z - p_x - p_y) \tag{4}$$

$$\dot{p}_x = X(Y + Z - X) \qquad \dot{p}_y = Y(Z + X - Y) \qquad \dot{p}_z = Z(X + Y - Z). \tag{5}$$

(Note the slight difference in definition, as regards to p_x, p_y, p_z, from our previous paper [3].) Equation (2) means that the energy has the particular value zero:

$$E = p_x^2 + p_y^2 + p_z^2 - 2p_x p_y - 2p_y p_z - 2p_z p_x + X^2 + Y^2 + Z^2 - 2XY - 2YZ - 2ZX = 0. \tag{6}$$

The ARS singularity analysis for equations (4) and (5) has yielded two different singular behaviours [3].

(i) X and p_x alone diverge while Y, Z, p_y, p_z are finite (or any other circular permutation):

$$X = \pm\frac{\mathrm{i}}{\tau - \tau_0} \qquad p_x = -\frac{1}{\tau - \tau_0}$$
$$Y = y_{01}(\tau - \tau_0) \qquad Z = z_{01}(\tau - \tau_0) \qquad p_y = p_2 \qquad p_z = p_3. \tag{7}$$

The resonances in this case are: $r : -1, 0, 0, 0, 0, 2$.

The resonance -1 is related, as usual, to the freedom of the location τ_0 of the singularity, while the quadruple 0 resonance is related to the free y_{01}, z_{01}, p_2, p_3 parameters. We have also verified that the $r = 2$ resonance indeed satisfies the compatibility condition. Thus, this expansion is generic, i.e. it has six free parameters and it is of Painlevé type.

(ii) All X, Y, Z, p_x, p_y, p_z diverge as simple poles.

$$X, Y, Z = \pm\frac{\mathrm{i}}{\tau - \tau_0} \qquad p_x, p_y, p_z = \frac{1}{\tau - \tau_0}. \tag{8}$$

The resonances in this case are $r: -1, -1, -1, 2, 2, 2$.

This non-generic case is intriguing since it possesses a triple (-1) resonance, a feature that might indicate a dominant logarithmic singularity. However, this turned out not to be the case and thus we concluded in [3] that type-(ii) singularities passed the ARS test. Our conclusion in view of these results was that the mixmaster universe model possesses the Painlevé property and is thus a candidate for integrability. However, a search of polynomial integrals of motion of low degrees yielded negative results and thus the dilemma persisted. At this point we were led back to the question: Is the model really integrable or not? An answer to this problem might be provided by the negative resonances.

Negative resonances have played a rather obscure role, at least in the modern tradition of Painlevé analysis through the use of the ARS algorithm. In fact the ARS recommendation concerning negative resonances is explicit [11]: 'ignore any roots (of the indicial equation $Q(r) = 0$) with $\mathrm{Re}(r) < 0$'. (In practice though, one asks for negative resonances to be integer valued too, although no compatibility at these resonances was ever considered.) This ARS view has been challenged recently by Kruskal [14] but we must point out that Kruskal has been drawing attention to this subtlety in his private exchanges with most Painlevé practitioners ever since the beginning of the modern singularity-analysis era. Kruskal, in fact, insists that a negative resonance may induce multivaluedness: directly if it is non-integer and through logarithmic terms if a compatibility condition is not satisfied at some integer negative resonance value. Kruskal's main point is that the Painlevé–Laurent series expansion can be considered as the lowest-order term of a perturbation series in the coefficient ϵ of the negative-resonance term

$$x(\tau) = \sum_{n=0}^{\infty} \epsilon^n x_n(\tau) \tag{9}$$

where $x_0(\tau)$ is the Painlevé series and the higher $x_n(\tau)$ are generalized power series determined successively for $n = 1, 2, \ldots$ to satisfy the differential equation. This ϵ-series should be valid for small ϵ and small, but not too small, $\tau - \tau_0$, i.e. in an annulus around τ_0 with inner radius depending on ϵ.

Kruskal's ideas were followed up by Conte, Fordy and Pickering [15, 16] and implemented algorithmically. They have shown that there exist cases (Chazy's equations being prominent among these) where the negative resonances play an important role in (non-)integrability. In order to follow Kruskal's approach, we consider the type-(ii) singular expansion as part of a perturbation expansion (in the coefficient ϵ of the negative resonance terms). In order to simplify the presentation we introduce a second small parameter η related to the positive resonances and thus propose the following expansion for X:

$$\begin{aligned} X = \pm\frac{\mathrm{i}}{\tau - \tau_0}\bigg\{ & 1 + \eta x_{01}(\tau - \tau_0)^2 + \eta^2 x_{02}(\tau - \tau_0)^4 + \cdots \\ & + \frac{\epsilon}{\tau - \tau_0}\left(x_{10} + \eta x_{11}(\tau - \tau_0)^2 + \eta^2 x_{12}(\tau - \tau_0)^4 + \cdots\right) \\ & + \frac{\epsilon^2}{(\tau - \tau_0)^2}\left(x_{20} + \eta x_{21}(\tau - \tau_0)^2 + \cdots\right)\bigg\} \end{aligned} \tag{10}$$

and similarly for Y, Z, p_x, p_y, p_z. The x_{10}, y_{10}, z_{10} and x_{01}, y_{01}, z_{01} are free parameters because -1 and 2 are triple resonances: the corresponding coefficients of the p_i's are determined from those of the X, Y, Z. By substituting the expansion (10) in the equations of motion (4) and (5), we can compute the coefficients order by order in ϵ. However, now we get a resonance condition at every order whenever the power of $\tau - \tau_0$ is -1 or $+2$. This means that a resonance condition will occur at the $\epsilon^n\eta^m$ terms whenever $n = 2m + 1$ or $n = 2m - 2$. We have started by checking the first two occurrences, namely, $m = 1$, $n = 3$ and $m = 2$, $n = 2$. Neither is automatically satisfied but necessitate some resonance compatibility condition to hold. At $m = 1$, $n = 3$, we find

$$(y_{10} - z_{10})K_1 = 0 \qquad (z_{10} - x_{10})K_2 = 0 \qquad (x_{10} - y_{10})K_3 = 0 \tag{11}$$

and

$$\begin{aligned} K_1 = {} & 2x_{01}(y_{10} - z_{10})(2x_{10} - y_{10} - z_{10}) - y_{01}(z_{10} - x_{10})(2z_{10} - x_{10} - y_{10}) \\ & - z_{01}(x_{10} - y_{10})(2y_{10} - x_{10} - z_{10}) \end{aligned} \tag{12}$$

and circular permutations for K_2, K_3. Clearly, $x_{10} = y_{10} = z_{10}$ is a possible way of satisfying this condition. A look at the expansion of X, Y, Z in this case indicates the true nature of this condition: it corresponds to taking an arbitrary shift in the singularity location τ_0, but *the same shift for all X's*, and expanding around it. Indeed, we have checked that this condition suffices in order to satisfy the higher-order conditions. In particular, the $m = 2, n = 2$ condition is satisfied with this choice of constraint. However, there is another possibility which satisfies the set of constraints (11): putting the K factors to zero. We find in this case that $x_{01} = y_{01} = z_{01}$. The physical interpretation of this constraint is not clear. We have also checked the implications of this choice on the constraints at the $m = 2, n = 2$ level. We found that the constraints were satisfied iff $x_{01} = y_{01} = z_{01}$. However, it was not clear whether this constraint was sufficient at all orders. Thus, we pushed our calculations one step beyond to orders $\epsilon^5\eta^2$ and $\epsilon^4\eta^3$. Order $\epsilon^4\eta^3$ is satisfied with this choice (and also with $x_{10} = y_{10} = z_{10}$). However, order $\epsilon^5\eta^2$ necessitates the introduction of new constraints (or just $x_{10} = y_{10} = z_{10}$ again) and thus the solution with the constraints $K_i = 0$ is insufficient.

So, although in the classical ARS approach the mixmaster universe model has a valid Painlevé expansion, the perturbative singular expansion (10) does not satisfy the compatibility condition at every order (unless a special expansion $x_{10} = y_{10} = z_{10}$ is considered). Now, what does this mean? This is far from being clear at the present stage. Already, the standard Painlevé conjecture has not yet acquired theorem status. The perturbative Painlevé approach is even harder to justify. An expansion such as (10) is, in a sense, asymptotic as $\epsilon \to 0$ to well behaved solutions. However, for finite ϵ, such an expansion may not converge to anything and it is asymptotic neither as $\tau \to \infty$ nor as $\tau - \tau_0 \to 0$. Still, the presence of logarithms (due to the non-satisfaction of the resonance compatibility conditions) may indicate non-integrability. In the work of Conte *et al* [16] it is precisely this approach that allows one to single out the integrable Chazy equations. As a further check, we have examined a paradigm of an integrable system: the integrable case of the Hénon–Heiles potential $V = y^3 + 2x^2y$ [17], using the perturbative Painlevé approach. Contrary to the case of the mixmaster universe model, all the negative-resonance conditions that we examined were satisfied. Thus, if the perturbative Painlevé approach is a better criterion of (non-)integrability than the 'classical' Painlevé approach then we have an indication that the mixmaster universe model may, after all, be non-integrable. In any case, this is the first *physical* model for which the negative resonances may play an important role: indeed this model passes the classical Painlevé test but not the perturbative one. In such a case, the non-integrability would be mediated by terms that are absent in standard asymptotic expansions of the solutions and this might be an explanation for the absence of large-scale chaotic behaviour.

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Note added in proof. After the present work was submitted, Cotsakis and Leach published a paper (1994 *J. Phys. A: Math. Gen.* 27 1625) presenting a Painlevé analysis of the mixmaster universe model. Their paper missed the main leading behaviour (case I of our paper [3]), namely, the one that has all six free parameters. Thus, the authors could not prove that the system had the Painlevé property.

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